Solutions to The Nonhomogeneous Chebyshev's Equation by Means of N-Fractional Calculus Operator (Conditions for Univalency of Functions and Applications)

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数理解析研究所講究録
第 1772 巻 2011 年 64-76

Solutions to The Nonhomogeneous Chebyshev's Equation by Means of N- Fractional Calculus Operator

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Abstract

In this article, solutions to the nonhomogeneous Chebyshev's equations

$$L[\varphi;z;\nu]=\varphi_2\cdot(z^2-1)+\varphi_1\cdot z-\varphi\cdot\nu^2=f \quad (z^2-1\neq 0, f\neq 0)$$

$$(\varphi_\alpha=d^\alpha\varphi/dz^\alpha \text{ for } \alpha>0,\ \varphi_0=\varphi=\varphi(z),\ f=f(z))$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i) $\varphi=(G(\nu)\cdot H(\nu))_{-(1+\nu)}\equiv\varphi^*_{[1](z,\nu)}$ (denote)

(ii) $\varphi=(H(\nu)\cdot G(\nu))_{-(1+\nu)}\equiv\varphi^*_{[2](z,\nu)}$

(iII) $\varphi=(G(-\nu)\cdot H(-\nu))_{\nu-1}\equiv\varphi^*_{[3](z,\nu)}$

(iv) $\varphi=(H(-\nu)\cdot G(-\nu))_{\nu-1}\equiv\varphi^*_{[4](z,\nu)}$

where

$$G(\nu)=(f_\nu\cdot(z^2-1)^{\nu-1/2})_{-1},\qquad H(\nu)=(z^2-1)^{-(\nu+1/2)}.$$

§ 0. Introduction (Definition of Fractional Calculus)

and

§ 1. Preliminary

are ommited, then refer to the previous paper for the Homogeneous Chebyshev's equation.

§ 2. Solutions to The Nonhomogeneous Chebyshev's Equations by Means of N-Fractional Calculus Operator

Theorem 1. *Let* $\varphi = \varphi(z) \in F$ and $f = f(z) \in F$, *then the nonhomogeneous Chebyshev's equation*

$$L[\varphi ; z ; v] = \varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi \cdot v^2 = f , \quad (v \in R,\ z^2 - 1 \neq 0) \tag{1}$$

has particular solutions of the forms (fractional differintegrated form);

Group I.

(i) $$\varphi(z) = (G(v) \cdot H(v))_{-(1+v)} \equiv \varphi^*_{[1](z,v)} \quad \text{(denote)} \tag{2}$$

(ii) $$\varphi(z) = (H(v) \cdot G(v))_{-(1+v)} \equiv \varphi^*_{[2](z,v)} \tag{3}$$

(iii) $$\varphi(z) = (G(-v) \cdot H(-v))_{v-1} \equiv \varphi^*_{[3](z,v)} \tag{4}$$

(iv) $$\varphi(z) = (H(-v) \cdot G(-v))_{v-1} \equiv \varphi^*_{[4](z,v)} \tag{5}$$

where

$$G(v) = (f_v \cdot (z^2 - 1)^{v-1/2})_{-1} , \qquad H(v) = (z^2 - 1)^{-(v+1/2)} . \tag{6}$$

Group II.

(i) $$\varphi(z) = (z^2 - 1)^{1/2} (P(v) \cdot H(v))_{-v} \equiv \varphi^*_{[5](z,v)} \tag{7}$$

(ii) $$\varphi(z) = (z^2 - 1)^{1/2} (H(v) \cdot P(v))_{-v} \equiv \varphi^*_{[6](z,v)} \tag{8}$$

(iii) $$\varphi(z) = (z^2 - 1)^{1/2} (P(-v) \cdot H(-v))_{v} \equiv \varphi^*_{[7](z,v)} \tag{9}$$

(iv) $$\varphi(z) = (z^2 - 1)^{1/2} (H(-v) \cdot P(-v))_{v} \equiv \varphi^*_{[8](z,v)} \tag{10}$$

where

$$P(\nu)=((f\cdot(z^2-1)^{-1/2})_{\nu-1}\cdot(z^2-1)^{\nu-1/2})_{-1}\,,\qquad H(\nu)=(z^2-1)^{-(\nu+1/2)}\quad ..\ (11)$$

Group III.

(i) $$\varphi(z)=(z-1)^{1/2}(Q(\nu)\cdot S(\nu))_{-(\nu+1/2)}\equiv\varphi^*{}_{[9](z,\nu)} \qquad (12)$$

(ii) $$\varphi(z)=(z-1)^{1/2}(S(\nu)\cdot Q(\nu))_{-(\nu+1/2)}\equiv\varphi^*{}_{[10](z,\nu)} \qquad (13)$$

(iii) $$\varphi(z)=(z-1)^{1/2}(Q(-\nu)\cdot S(-\nu))_{\nu-1/2}\equiv\varphi^*{}_{[11](z,\nu)} \qquad (14)$$

(iv) $$\varphi(z)=(z-1)^{1/2}(S(-\nu)\cdot Q(-\nu))_{\nu-1/2}\equiv\varphi^*{}_{[12](z,\nu)} \qquad (15)$$

where

$$Q(\nu)=((f\cdot(z-1)^{-1/2})_{\nu-1/2}\cdot(z-1)^{\nu}(z+1)^{\nu-1})_{-1}\,,\quad S(\nu)=((z-1)^{-(\nu+1)}(z+1)^{-\nu})\,. \quad (16)$$

Group IV.

(i) $$\varphi(z)=(z+1)^{1/2}(T(\nu)\cdot Y(\nu))_{-(\nu+1/2)}\equiv\varphi^*{}_{[13](z,\nu)} \qquad (17)$$

(ii) $$\varphi(z)=(z+1)^{1/2}(Y(\nu)\cdot T(\nu))_{-(\nu+1/2)}\equiv\varphi^*{}_{[14](z,\nu)} \qquad (18)$$

(iii) $$\varphi(z)=(z+1)^{1/2}(T(-\nu)\cdot Y(-\nu))_{\nu-1/2}\equiv\varphi^*{}_{[15](z,\nu)} \qquad (19)$$

(iv) $$\varphi(z)=(z+1)^{1/2}(Y(-\nu)\cdot T(-\nu))_{\nu-1/2}\equiv\varphi^*{}_{[16](z,\nu)} \qquad (20)$$

where

$$T(\nu)=((f\cdot(z+1)^{-1/2})_{\nu-1/2}\cdot(z-1)^{\nu-1}(z+1)^{\nu})_{-1}\,,\quad Y(\nu)=((z-1)^{-\nu}(z+1)^{-(\nu+1)})\,. \quad (21)$$

Proof of Group I ;

Operate N^{α} to the both sides of (1), we have then

$$\varphi_{2+\alpha}\cdot(z^2-1)+\varphi_{1+\alpha}\cdot z(2\alpha+1)+\varphi_{\alpha}\cdot(\alpha^2-\nu^2)=f_{\alpha}\,,\quad (f_{\alpha}\neq 0)\,. \qquad (22)$$

(Refer to the proof of **Group I** in § **2.** in the previous paper for Homogeneous one.)

(I) Case $\alpha=\nu$;

In this case we obtain

$$\varphi_{2+\nu}\cdot(z^2-1)+\varphi_{1+\nu}\cdot z(2\nu+1)=f_{\nu}\,, \qquad (23)$$

from (22), then letting

$$\varphi_{1+\nu}=\psi=\psi(z)\qquad(\varphi=\psi_{-(1+\nu)})\,, \qquad (24)$$

yields

$$\psi_1\cdot(z^2-1)+\psi\cdot z(2\nu+1)=f_{\nu}\,, \qquad (25)$$

from (24). A particular solution to this linear first order equation is given by

$$\psi=(f_{\nu}\cdot(z^2-1)^{\nu-1/2})_{-1}\cdot(z^2-1)^{-(\nu+1/2)}\,. \qquad (26)$$

Therefore, we obtain

$$\varphi = \psi_{-(1+\nu)} = ((f_\nu \cdot (z^2-1)^{\nu-12})_{-1} \cdot (z^2-1)^{-(\nu+1/2)})_{-(1+\nu)} \cdot \qquad (27)$$

$$= (G(\nu) \cdot H(\nu))_{-(1+\nu)} \equiv \varphi^*_{[1](z,\nu)} \qquad (2)$$

from (24) and (26).

Inversely, we have

$$\varphi_{2+\nu} = \psi_1 = (f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}}) \cdot (z^2-1)^{-(\nu+1/2)}$$
$$- (f_\nu \cdot (z^2-1)^{\nu-1/2})_{-1} \cdot (\nu+\tfrac{1}{2}) \cdot 2z(z^2-1)^{-(\nu+1/2+1)} \qquad (28)$$

and

$$\varphi_{1+\nu} = \psi = (f_\nu \cdot (z^2-1)^{\nu-1/2})_{-1} \cdot (z^2-1)^{-(\nu+1/2)} \qquad (29)$$

from (2) respectively. Then we obtain

$$\text{LHS of } (23) = \{(f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}}) \cdot (z^2-1)^{-(\nu+\frac{1}{2})}$$
$$- (f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}})_{-1} \cdot (\nu+\tfrac{1}{2}) \cdot 2z(z^2-1)^{-(\nu+\frac{1}{2}+1)}\}(z^2-1)$$
$$+z(2\nu+1)\ (f_\nu \cdot (z^2-1)^{\nu-\frac{1}{2}})_{-1} \cdot (z^2-1)^{-(\nu+\frac{1}{2})} = f_\nu \qquad (30)$$

applying (28) and (29).

That is, the function shown by (2) satisfies equation (23).

Moreover, we have (1) operating $N^{-\nu}$ to the both sides of (23).

Therefore, the function given by (2) satisfies equation (1).

Next changing the order $G(\nu)$ and $H(\nu)$ in parenthesis $(\quad\cdot\quad)_{-(1+\nu)}$ in (2) we obtain

$$\varphi = (H(\nu) \cdot G(\nu))_{-(1+\nu)} \equiv \varphi^*_{[2](z,\nu)} \cdot \qquad (3)$$

where

$$\varphi^*_{[1](z,\nu)} \neq \varphi^*_{[2](z,\nu)} \quad (\text{for } -(1+\nu) \notin \mathbf{Z}^+_0) \ . \qquad (31)$$

(II) Case $\alpha = -\nu$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (2) and (3), we obtain

$$\varphi = (G(-\nu) \cdot H(-\nu))_{\nu-1} \equiv \varphi^*_{[3](z,\nu)} \qquad (4)$$

$$\varphi = (H(-\nu) \cdot G(-\nu))_{\nu-1} \equiv \varphi^*_{[4](z,\nu)} \cdot \qquad (5)$$

where

$$\varphi^*_{[3](z,\nu)} \neq \varphi^*_{[4](z,\nu)} \quad (\text{for } (\nu-1) \notin \mathbf{Z}^+_0) \ . \qquad (32)$$

And

$$\varphi^*_{[3](z,\nu)} = \varphi^*_{[1](z,-\nu)}, \qquad \varphi^*_{[4](z,\nu)} = \varphi^*_{[2](z,-\nu)} \ . \qquad (33)$$

Note 1. When $f_v = 0$, we have

$$\varphi^*_{[1](z,v)} = ((0)_{-1} \cdot (z^2-1)^{-(v+1/2)})_{-(1+v)} = (K \cdot (z^2-1)^{-(v+1/2)})_{-(1+v)} = 0 \ . \quad (34\text{-}1)$$

(for $(1+v) \notin \mathbf{Z}_0^+$)

(K; arbitrary constant for integation)

$$\varphi^*_{[2](z,v)} = ((z^2-1)^{-(v+1/2)} \cdot (0)_{-1})_{-(1+v)} = ((z^2-1)^{-(v+1/2)} \cdot K)_{-(1+v)}$$

$$= ((z^2-1)^{-(v+1/2)})_{-(1+v)} K \quad (34\text{-}2)$$

from (.2) and (3) , respectively, by Lemmas (i v) and (i).

And we have

$$((z^2-1)^{-(v+1/2)} K)_{-(1+v)} = (K(z^2-1)^{-(v+1/2)})_{-(1+v)} = K((z^2-1)^{-(v+1/2)})_{-(1+v)} \quad (34\text{-}3)$$

by our definition **§ 0.** (1) , for N- Fractional Calculus.

Proof of Group I I ;

Set

$$\varphi = (z^2-1)^{\lambda} \phi, \qquad \phi = \phi(z), \quad (35)$$

we have then

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot z(4\lambda+1) + \phi \cdot \{(4\lambda^2 - v^2) + \frac{2\lambda(2\lambda-1)}{z^2-1}\} = f \cdot (z^2-1)^{-\lambda} \quad (36)$$

from (1), applying (35).

(Refer to the proof of **Group II** in § **2.** in the previous paper for Homogeneous one.)

When $\lambda = 0$, (36) is reduced to (1). We have then the same particular solutions as Group I

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot 3z + \phi \cdot (1-v^2) = f \cdot (z^2-1)^{-1/2} \quad (37)$$

from (36).

Operate N^{α} to the both sides of (37), then yields

$$\phi_{2+\alpha} \cdot (z^2-1) + \phi_{1+\alpha} \cdot z(2\alpha+3) + \phi_{\alpha} \cdot ((\alpha+1)^2 - v^2) = (f \cdot (z^2-1)^{-1/2})_{\alpha} \ . \quad (38)$$

(I) Case $\alpha = v-1$;

In this case ,letting

$$\phi_v = V = V(z) \qquad (\phi = V_{-v}), \quad (39)$$

we obtain

$$V_1 \cdot (z^2-1) + V \cdot z(2\nu+1) = (f \cdot (z^2-1)^{-1/2})_{\nu-1} \quad , \tag{40}$$

from (38). A particular solution to this linear first order equation is given by

$$V = ((f \cdot (z^2-1)^{-1/2})_{\nu-1} \cdot (z^2-1)^{\nu-1/2})_{-1} \cdot (z^2-1)^{-(\nu+1/2)} = P(\nu)H(\nu) \quad . \tag{41}$$

Therefore, we obtain

$$\varphi = (z^2-1)^{1/2}\Big(((f \cdot (z^2-1)^{-1/2})_{\nu-1} \cdot (z^2-1)^{\nu-1/2})_{-1} \cdot (z^2-1)^{-(\nu+1/2)}\Big)_{-\nu} \tag{42}$$

$$= (z^2-1)^{1/2}(P(\nu) \cdot H(\nu))_{-\nu} \equiv \varphi^*_{[5](z,\nu)} \quad , \tag{7}$$

from (35), applying (39) and (41), for $\lambda = 1/2$.

Next changing the order $P(\nu)$ and $H(\nu)$ in parenthesis $(\quad \cdot \quad)_{-\nu}$ in (7) we obtain

$$\varphi = (z^2-1)^{1/2}(H(\nu) \cdot P(\nu))_{-\nu} \equiv \varphi^*_{[6](z,\nu)} \quad . \tag{8}$$

where

$$\varphi^*_{[5](z,\nu)} \neq \varphi^*_{[6](z,\nu)} \quad (\text{for} \quad (-\nu) \notin \boldsymbol{Z}_0^+) \quad . \tag{43}$$

(I I) Case $\alpha = -\nu - 1$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (7) and (8), we obtain

$$\varphi = (z^2-1)^{1/2}(P(-\nu) \cdot H(-\nu))_{\nu} \equiv \varphi^*_{[7](z,\nu)} \quad , \tag{9}$$

$$\varphi = (z^2-1)^{1/2}(H(-\nu) \cdot P(-\nu))_{\nu} \equiv \varphi^*_{[8](z,\nu)} \quad . \tag{10}$$

where

$$\varphi^*_{[7](z,\nu)} \neq \varphi^*_{[8](z,\nu)} \quad (\text{for} \quad \nu \notin \boldsymbol{Z}_0^+) \quad . \tag{44}$$

And

$$\varphi^*_{[7](z,\nu)} = \varphi^*_{[5](z,-\nu)} , \qquad \varphi^*_{[8](z,\nu)} = \varphi^*_{[6](z,-\nu)} \quad . \tag{45}$$

Proof of Group I I I ;

Set

$$\varphi = (z-1)^{\lambda}\phi, \qquad \phi = \phi(z), \tag{46}$$

we have then

$$\phi_2 \cdot (z^2-1) + \phi_1 \cdot \{z(2\lambda+1) + 2\lambda\} + \phi \cdot \{(\lambda^2 - \nu^2) + \frac{\lambda(2\lambda-1)}{z-1}\} = f \cdot (z-1)^{-\lambda} \tag{47}$$

from (1), applying (46)

(Refer to the proof of **Group III** in § **2.** in the previous paper for Homogeneous one)

When $\lambda = 0$, (47) is reduced to (1). We have then the same particular solutions as Group I .

When $\lambda = 1/2$, we have

$$\phi_2 \cdot (z^2 - 1) + \phi_1 \cdot (2z + 1) + \phi \cdot (1/4 - \nu^2) = f \cdot (z-1)^{-1/2} \qquad (48)$$

from (47).

Operate N^α to the both sides of (48), then yields

$$\phi_{2+\alpha} \cdot (z^2 - 1) + \phi_{1+\alpha} \cdot \{z(2\alpha + 2) + 1\} + \phi_\alpha \cdot \{(\alpha + 1/2)^2 - \nu^2\} = (f \cdot (z-1)^{-1/2})_\alpha \ . \qquad (49)$$

(I) Case $\alpha = \nu - 1/2$;

In this case letting

$$\phi_{\nu+1/2} = V = V(z) \qquad (\phi = V_{-(\nu+1/2)}), \qquad (50)$$

we obtain

$$V_1 \cdot (z^2 - 1) + V \cdot \{z(2\nu + 1) + 1\} = (f \cdot (z-1)^{-1/2})_{\nu-1/2} \ , \qquad (51)$$

from (49).

A particular solution to this equation is given by

$$V = ((f \cdot (z-1)^{-1/2})_{\nu-1/2} \cdot \frac{z-1}{(z^2-1)^{1-\nu}})_{-1} \cdot \frac{(z^2-1)^{-\nu}}{z-1} = Q(\nu)S(\nu) \ . \qquad (52)$$

Therefore, we obtain

$$\varphi = (z-1)^{1/2}\Big(((f \cdot (z-1)^{-1/2})_{\nu-1/2} \cdot (z-1)^\nu (z+1)^{\nu-1})_{-1} \cdot ((z-1)^{-(\nu+1)}(z+1)^{-\nu})\Big)_{-(\nu+1/2)} \qquad (53)$$

$$= (z-1)^{1/2}(Q(\nu) \cdot S(\nu))_{-(\nu+1/2)} \equiv \varphi^*_{[9](z,\nu)} \ , \qquad (12)$$

from (46) and (50), applying (52), for $\lambda = 1/2$.

Next changing the order $Q(\nu)$ and $S(\nu)$ in parenthesis $(\quad \cdot \quad)_{-(\nu+1/2)}$ in (12) we obtain

$$\varphi = (z-1)^{1/2}(S(\nu) \cdot Q(\nu))_{-\nu} \equiv \varphi^*_{[10](z,\nu)} \ . \qquad (13)$$

where

$$\varphi^*_{[9](z,\nu)} \neq \varphi^*_{[10](z,\nu)} \quad (\text{for } -(\nu + 1/2) \notin \mathbf{Z}_0^+) \ . \qquad (54)$$

(II) Case $\alpha = -\nu - 1/2$;

In the same way as **(I)** above, setting $-\nu$ instead of ν in the solutions (12) and (13), we obtain

$$\varphi = (z-1)^{1/2}(Q(-\nu) \cdot S(-\nu))_\nu \equiv \varphi^*_{[11](z,\nu)} \ , \qquad (14)$$

$$\varphi = (z-1)^{1/2}(S(-\nu)\cdot Q(-\nu))_\nu \equiv \varphi^*_{[12](z,\nu)} \quad . \tag{15}$$

where

$$\varphi^*_{[11](z,\nu)} \neq \varphi^*_{[12](z,\nu)} \quad (\text{for } (\nu - 1/2) \notin \mathbf{Z}_0^+) \ . \tag{55}$$

And

$$\varphi^*_{[11](z,\nu)} = \varphi^*_{[9](z,-\nu)}, \qquad \varphi^*_{[12](z,\nu)} = \varphi^*_{[10](z,-\nu)} \quad . \tag{56}$$

Proof of Group I V ;

Set

$$\varphi = (z+1)^\lambda \phi, \qquad \phi = \phi(z), \tag{57}$$

we have then

$$\phi_2\cdot(z^2-1) + \phi_1\cdot\{z(2\lambda+1) - 2\lambda\} + \phi\cdot\{(\lambda^2 - \nu^2) - \frac{\lambda(2\lambda-1)}{z+1}\} = f\cdot(z+1)^{-\lambda} \tag{58}$$

from (1) applying (57).

(Refer to the proof of **Group IV** in § **2.** in the previous paper for Homogeneous one)

When $\lambda = 0$, (58) is reduced to (1). We have then the same particular solutions as Group I .

When $\lambda = 1/2$, we have

$$\phi_2\cdot(z^2-1) + \phi_1\cdot(2z-1) + \phi\cdot(1/4 - \nu^2) = f\cdot(z+1)^{-1/2} \tag{59}$$

from (58).

Operate N^α to the both sides of (59), then yields

$$\phi_{2+\alpha}\cdot(z^2-1) + \phi_{1+\alpha}\cdot\{z(2\alpha+2) - 1\} + \phi_\alpha\cdot\{(\alpha+1/2)^2 - \nu^2\} = (f\cdot(z+1)^{-1/2})_\alpha \ . \tag{60}$$

(I) Case $\alpha = \nu - 1/2$;

Letting

$$\phi_{\nu+1/2} = V = V(z) \qquad (\phi = V_{-(\nu+1/2)}), \tag{61}$$

we obtain

$$V_1\cdot(z^2-1) + V\cdot\{z(2\nu+1) - 1\} = (f\cdot(z+1)^{-1/2})_{\nu-1/2} \quad , \tag{62}$$

from (60).

A particular solution to this equation is given by

$$V = ((f\cdot(z+1)^{-1/2})_{\nu-1/2}\cdot(z-1)^{\nu-1}(z+1)^\nu)_{-1}\cdot(z-1)^{-\nu}(z+1)^{-(\nu+1)} = T(\nu)\cdot Y(\nu) \ . \tag{63}$$

Therefore, we obtain

$$\varphi = (z+1)^{1/2}\Big(((f\cdot(z+1)^{-1/2})_{\nu-1/2}\cdot(z-1)^{\nu-1}(z+1)^\nu)_{-1}\cdot((z-1)^{-\nu}(z+1)^{-(\nu+1)})\Big)_{-(\nu+1/2)} \tag{64}$$

$$= (z+1)^{1/2}(T(v)\cdot Y(v))_{-(v+1/2)} \equiv \varphi^*_{[13](z,v)} \quad , \tag{17}$$

from (57) and (61), applying (63), for $\lambda = 1/2$.

Next changing the order $T(v)$ and $Y(v)$ in parenthesis $(\quad\cdot\quad)_{-(v+1/2)}$ in (17) we obtain

$$\varphi = (z+1)^{1/2}(Y(v)\cdot T(v))_{-(v+1/2)} \equiv \varphi^*_{[14](z,v)} \quad . \tag{18}$$

where

$$\varphi^*_{[13](z,v)} \neq \varphi^*_{[14](z,v)} \quad (\text{for } -(v+1/2) \notin \boldsymbol{Z}_0^+) \ . \tag{65}$$

(I I) Case $\alpha = -v - 1/2$;

In the same way as **(I)** above, setting $-v$ instead of v in the solutions (17) and (18), we obtain

$$\varphi = (z+1)^{1/2}(T(-v)\cdot Y(-v))_{v-1/2} \equiv \varphi^*_{[15](z,v)} \quad , \tag{19}$$

$$\varphi = (z+1)^{1/2}(Y(-v)\cdot T(-v))_{v-1/2} \equiv \varphi^*_{[16](z,v)} \quad . \tag{20}$$

where

$$\varphi^*_{[15](z,v)} \neq \varphi^*_{[16](z,v)} \quad (\text{for } (v-1/2) \notin \boldsymbol{Z}_0^+) \ . \tag{66}$$

And

$$\varphi^*_{[15](z,v)} = \varphi^*_{[13](z,-v)} \ , \qquad \varphi^*_{[16](z,v)} = \varphi^*_{[14](z,-v)} \quad . \tag{67}$$

§3. Some Example

(i) When $v = -1$ and $f = (z-1)^{-1}$ we have

$$\varphi_2 \cdot (z^2 - 1) + \varphi_1 \cdot z - \varphi = (z-1)^{-1} \tag{15}$$

and

$$\varphi = \varphi^*_{[1](z,-1)} = \left((f_{-1}\cdot(z^2-1)^{-3/2})_{-1}(z^2-1)^{1/2}\right)_0 \tag{16}$$

$$= \left(\log(z-1)\cdot(z^2-1)^{-3/2}\right)_{-1}(z^2-1)^{1/2} \tag{17}$$

from **§2.** (1) and **§2.** (2) respectively.

Hence

$$\varphi_1 = \log(z-1)\cdot(z^2-1)^{-1} + \left(\log(z-1)\cdot(z^2-1)^{-3/2}\right)_{-1}(z^2-1)^{-1/2}z \tag{18}$$

and

$$\varphi_2 = (z-1)^{-1}(z^2-1)^{-1} - \log(z-1)\cdot 2z(z^2-1)^{-2} + \log(z-1)\cdot z(z^2-1)^{-2}$$
$$+\left(\log(z-1)\cdot(z^2-1)^{-3/2}\right)_{-1}\{-z^2(z^2-1)^{-3/2}+(z^2-1)^{-1/2}\} \tag{19}$$

respectively.

Then applying (17), (18) and (19), we obtain

$$\text{LHS of } (15) = (z-1)^{-1} \quad . \tag{20}$$

The function shown by (17) satisfies equation (15) clearly.

(i i) When $\nu = -1/2$ and $f = (z-1)^{-1/2}$ we have

$$\varphi_2\cdot(z^2-1)+\varphi_1\cdot z-\varphi\cdot(1/4) = (z-1)^{-1/2} \tag{21}$$

and

$$\varphi = \varphi^*{}_{[9](z,-1/2)} = (z+1)^{1/2}\left(\log(z-1)\cdot(z-1)^{-1/2}(z+1)^{-3/2}\right)\Big)_{-1} \tag{22}$$

from §2. (1) and §2. (9) respectively.

Hence we have

$$\varphi_1 = \tfrac{1}{2}(z+1)^{-1/2}(\,W\,)_{-1} + \log(z-1)\cdot(z-1)^{-1/2}(z+1)^{-1} \tag{23}$$
$$(W = \log(z-1)\cdot(z-1)^{-1/2}(z+1)^{-3/2}\ )$$

and

$$\varphi_2 = -\tfrac{1}{4}(z+1)^{-3/2\,1}(W)_{-1} + (z-1)^{-3/2}(z+1)^{-1} + \{\tfrac{1}{2}(z-1)^{-1/2}(z+1)^{-2}$$
$$-\tfrac{1}{2}(z-1)^{-3/2}(z+1)^{-1} - (z-1)^{-1/2}(z+1)^{-2}\}\log(z-1) \tag{24}$$

from (22), respectively.

Then applying (22), (23) and (24), we obtain

$$\text{LHS of } (21) = (\,W\,)_{-1}(z+1)^{-1/2}\{-\tfrac{1}{4}(z-1)+\tfrac{1}{2}z-\tfrac{1}{4}(z+1)\}+(z-1)^{-1/2}$$
$$+(z+1)^{-1}(z-1)^{-1/2}\{\tfrac{1}{2}(z-1)-\tfrac{1}{2}(z+1)-(z-1)+z\}\log(z-1)$$
$$= (z-1)^{-1/2} \quad . \tag{25}$$

The function shown by (22) satisfies equation (21) clearly.

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